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Suppose $x, y, z > 0$ such that $x + y + z = 3$. Find the minimum value of

$$\frac{1}{x} + \frac{1}{xy} + \frac{1}{xyz}.$$

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First we will find $\min\left\{\frac{1}{x} + \frac{1}{xy} \mid x, y > 0 \text{ and } x + y = a\right\}$.

We have $\frac{1}{x} + \frac{1}{xy} = \frac{1}{a}\left(\frac{a}{x} + \frac{x+y}{xy}\right) = \frac{1}{a}\left(\frac{a+1}{x} + \frac{1}{y}\right) \geq \frac{1}{a}\frac{\left(\sqrt{a+1}+1\right)^2}{x+y} = \frac{\left(\sqrt{a+1}+1\right)^2}{a^2}$, with equality iff $y = \frac{x}{\sqrt{a+1}}$ and $x + y = a$, that is iff $x = \frac{a\sqrt{a+1}}{\sqrt{a+1}+1} = \sqrt{a+1}\left(\sqrt{a+1}-1\right) = a + 1 - \sqrt{a+1}$ and $y = \sqrt{a+1} - 1$.

Thus, $\min\left\{\frac{1}{x} + \frac{1}{xy} \mid x, y > 0 \text{ and } x + y = a\right\} = \frac{\left(\sqrt{a+1}+1\right)^2}{a^2} = \frac{1}{\left(\sqrt{a+1}-1\right)^2}$.

Coming back to the problem, since $y + z = 3 - x$ and $\frac{1}{x} + \frac{1}{xy} + \frac{1}{xyz} = \frac{1}{x}\left(1 + \frac{1}{y} + \frac{1}{yz}\right)$ we obtain that $\min\left\{\frac{1}{y} + \frac{1}{yz} \mid y, z > 0 \text{ and } y + z = 3 - x\right\} = \frac{1}{\left(\sqrt{4-x}-1\right)^2}$

and, therefore, $\min\left\{\frac{1}{x} + \frac{1}{xy} + \frac{1}{xyz} \mid x, y, z > 0, y > 0 \text{ and } x + y + z = 3\right\} = \min_{0<x<3} \frac{1}{x}\left(1 + \frac{1}{\left(\sqrt{4-x}-1\right)^2}\right) = \min_{1<t<2} h(t)$, where $t := \sqrt{4-x} \in (1,2)$ and

$h(t) := \frac{1}{4-t^2}\left(1 + \frac{1}{(t-1)^2}\right)$.

Since $h'(t) = 0 \Leftrightarrow \frac{2(t^2-2t+4)(t^2-t-1)}{(t-1)^3(t+2)^2(t-2)^2} = 0 \Leftrightarrow t^2 - t - 1 = 0 \Leftrightarrow \left[\begin{array}{l} t = \frac{1+\sqrt{5}}{2} \\ t = \frac{1-\sqrt{5}}{2} \end{array}\right.$,

$h'(t) < 0$ if $t \in \left(1, \frac{1+\sqrt{5}}{2}\right)$ and $h'(t) > 0$ if $t \in \left(\frac{1+\sqrt{5}}{2}, 2\right)$ then

$$\min_{t\in(1,2)} h(t) = h\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1}{4 - \left(\frac{1+\sqrt{5}}{2}\right)^2}\left(1 + \frac{1}{\left(\frac{1+\sqrt{5}}{2} - 1\right)^2}\right) = \frac{3+\sqrt{5}}{2}.$$

Thus, $\min\left\{\frac{1}{x} + \frac{1}{xy} + \frac{1}{xyz} \mid x, y, z > 0, y > 0 \text{ and } x + y + z = 3\right\} = \frac{3+\sqrt{5}}{2}$ and attainable if $x = 4 - \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{5-\sqrt{5}}{2}$, $y = 4 - x - \sqrt{4-x} = 1$ and

$z = 3 - \left(\frac{5-\sqrt{5}}{2} + 1\right) = \frac{\sqrt{5}-1}{2}$.